MATH 5061 Problem Set 3[1]
Due date: Mar 18, 2020

Problems: (Please either type up your assignment or scan a copy of your written assignment into ONE PDF file and send it to me by email on/before the due date. Please remember to write down your name and SID. Questions marked with a † is optional.)

1. Let $F : M \to N$ be a smooth map and E be a vector bundle over N. Give the definition of the pullback bundle F^*E as a vector bundle over M. If D is a connection on E, define a pullback connection F^*D on F^*E. How is the curvature of F^*D related to that of D?

2. Let D be a connection on a vector bundle E over the base manifold M. We may extend D to a map $D : \Gamma(\wedge^k T^*M \otimes E) \to \Gamma(\wedge^{k+1} T^*M \otimes E)$ by setting
$$D(\omega \otimes s) := d\omega \otimes s + (-1)^k \omega \wedge Ds$$
where $\omega \in \Omega^k(M)$ and $s \in \Gamma(E)$.
   (a) Show that $D^2 s = \Omega s$ for any $s \in \Gamma(E)$, where Ω is the curvature of D viewed as an $\mathrm{End}(E)$-valued 2-form.
   (b) Show that $D^2 : \Gamma(\wedge^k T^*M \otimes E) \to \Gamma(\wedge^{k+2} T^*M \otimes E)$ satisfies $D^2\sigma = \Omega \wedge \sigma$.

3. (a) Given a connection D on a vector bundle E over M, we define the induced connection (also denoted by D) on the dual bundle E^* on M defined by the condition
$$X\big(s^*(s)\big) = (D_X s^*)(s) + s^*(D_X s)$$
for any $X \in \mathfrak{X}(M)$, $s \in \Gamma(E)$, $s^* \in \Gamma(E^*)$. Prove that it does define a connection on E^* and compute its curvature.
   (b) Let E and E' be vector bundles over the same base manifold M, equipped with connections D and D' respectively. Prove that the bundle $E \otimes E'$ has a natural connection ∇ determined by the condition
$$\nabla(s \otimes s') = Ds \otimes s' + s \otimes D's'$$
for any $s \in \Gamma(E)$, $s' \in \Gamma(E')$. What is the curvature of ∇?

4. Give a direct proof that a flat connection is locally trivial along the following line of argument: Let $P \in M$, and take local coordinates in a neighborhood of P so that P is the origin. It suffices to show that any nonzero element $s_0 \in E_P$ can be extended as a parallel section in a neighborhood. Do the extension first by parallel translation along the x_1-axis. Extend to the x_1x_2-plane by parallel translation along x_2 coordinate curves from points on the x_1 axis. Use the fact that covariant derivatives commute (since the connection is flat) to assert that this section is parallel in both the x_1 and x_2 directions. Now extend to the $x_1x_2x_3$ space by parallel translation along x_3 coordinate curves. Repeat until the section is defined in a cubical neighborhood of the origin.

5. (a) Show that a flat connection on a simply connected manifold is globally trivial; i.e., there is a global basis of parallel sections.
   (b) Let M be a non-simply connected compact manifold, and consider a base point $P \in M$. Let D be a flat connection on a vector bundle over M, and show that parallel transport around closed loops based at P depends only on the homotopy class of the loop and defines a homomorphism $\rho : \pi_1(M, P) \to GL(k, \mathbb{R})$. It is called the *holonomy representation of D*.
   (c) † Suppose we identify the fundamental group of M with a group Γ of deck transformations acting on the universal covering manifold $\tilde{M}$, and suppose we are given a homomorphism $\rho : \Gamma \to GL(k, \mathbb{R})$. We then form the trivial bundle $\tilde{E} = \tilde{M} \times \mathbb{R}^k$ with its natural trivial connection $\tilde{D}$. We define an action of Γ on $\tilde{E}$ by setting $\gamma(P, v) = (\gamma(P), \rho(\gamma)v)$. Show that $E = \tilde{E}/\Gamma$ is a flat bundle over M with holonomy ρ.

[1]Last revised on March 18, 2020